Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Linear Algebra and its Applications 413 (2006) 131–147
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

On the spectral radius of trees with fixed diameter

Ji-Ming Guo [a,b,*], Jia-Yu Shao [a,1]

[a] *Department of Applied Mathematics, Tongji University, Shanghai 200092, China*
[b] *Department of Mathematics, University of Petroleum, Shandong, Dongying 257061, China*

Received 20 January 2005; accepted 10 August 2005
Available online 18 October 2005
Submitted by R.A. Brualdi

Abstract

Let $T_{(n,d)}$ be the set of trees on n vertices with diameter d. In this paper, the first $\left\lfloor \frac{d}{2} \right\rfloor + 1$ spectral radii of trees in the set $T_{(n,d)}$ $(3 \leqslant d \leqslant n-4)$ are characterized.

AMS classification: 05C50

Keywords: Spectral radius; Tree; Diameter

1. Introduction

In this paper, we consider only connected finite graphs and, in particular, trees. Let $G = (V(G), E(G))$ be a graph on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Its adjacency matrix is defined to be the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i is adjacent to v_j; and $a_{ij} = 0$, otherwise. It follows immediately that if G is a simple graph, then $A(G)$ is a real symmetric $(0, 1)$ matrix in which every diagonal entry is zero, all of its eigenvalues are real. We assume, without loss of generality, that they are ordered in nonincreasing order, i.e.,

* Corresponding author.
E-mail addresses: gjm2248@sina.com (J.-M. Guo), jyshao@sh163.net (J.-Y. Shao).
[1] Research supported by National Natural Science Foundation of China 10331020.

0024-3795/$ - see front matter
doi:10.1016/j.laa.2005.08.008

$$\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G),$$

and call $\lambda_k(G)$ the kth largest eigenvalue of G. In particular, we call $\lambda_1(G)$ the spectral radius of G. The characteristic polynomial of G is just $\det(xI - A(G))$, which is denoted by $\Phi(G, x)$ or simply by $\Phi(G)$.

Recently, many authors studied the spectral radius of a tree with some given invariant, such as [4,9] etc. To classify and order graphs according to their eigenvalues is an interesting problem proposed by Cvetković in [2]. In [7], Hofmeister has determined the first five values of the spectral radius of trees with n vertices and the corresponding trees for these values. In [3], Chang and Huang extend this order to the eighth tree. In this paper, we give a new ordering for the trees with given diameter according to their spectral radius. Let $T_{(n,d)}$ be the set of trees on n vertices with diameter d. For the cases $3 \leqslant d \leqslant n-4$, the first $\lfloor \frac{d}{2} \rfloor + 1$ spectral radii of trees in the set $T_{(n,d)}$ are given, where $\lfloor \frac{d}{2} \rfloor$ denotes the largest integer no more than $\frac{d}{2}$. In addition, for the cases $d = n-3$ and $d = n-2$, the first $\lfloor \frac{d}{2} \rfloor - 1$ and first $\lfloor \frac{d}{2} \rfloor$ spectral radii of trees in the set $T_{(n,d)}$ are also given, respectively.

2. Preliminaries

In this section, we list some known results which will be used in this paper.

Lemma 2.1 [1]. *Let $e = uv$ be an edge of a tree T. Then the characteristic polynomial $\Phi(T)$ satisfies*

$$\Phi(T) = \Phi(T - e) - \Phi(T - u - v).$$

Lemma 2.2 [1]. *Let v be a vertex of a tree T. Then the characteristic polynomial $\Phi(T)$ satisfies*

$$\Phi(T) = x\Phi(T - v) - \sum_{w} \Phi(T - v - w),$$

where the summation extends over those vertices w adjacent to v.

The following three results are attributed to Li and Feng [8].

Lemma 2.3. *Let G be a connected graph, and let G' be a proper subgraph of G. Then*

$$\lambda_1(G) > \lambda_1(G'),$$

and for $x \geqslant \lambda_1(G)$,

$$\Phi(G', x) > \Phi(G, x).$$

Lemma 2.4. *Let v be a vertex in a graph G and suppose that two new paths P: $vv_1v_2 \cdots v_k$ and $Q : vu_1u_2 \cdots u_m$ of length k, m $(k \geqslant m \geqslant 1)$ are attached to G at*

v, respectively, to form a new graph $G_{k,m}$, where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_m$ are distinct new vertices. Then for $x > \lambda_1(G_{k,m})$, we have

$$\Phi(G_{k+1,m-1}, x) > \Phi(G_{k,m}, x).$$

In particular,

$$\lambda_1(G_{k,m}) > \lambda_1(G_{k+1,m-1}).$$

Lemma 2.5. *Let uv be an edge of a graph G, each of the vertices u and v has degree at least 2, and suppose that two new paths $P : uu_1u_2 \cdots u_k$ and $Q : vv_1v_2 \cdots v_m$ of length k, m $(k \geqslant m \geqslant 1)$ are attached to G at u and v, respectively, to form a new graph $M_{k,m}$, where $u_1, u_2, \ldots, u_k$ and $v_1, v_2, \ldots, v_m$ are distinct new vertices. Then for $x > \lambda_1(M_{k,m})$, we have*

$$\Phi(M_{k+1,m-1}, x) > \Phi(M_{k,m}, x).$$

In particular,

$$\lambda_1(M_{k,m}) > \lambda_1(M_{k+1,m-1}).$$

The following result can be found in [6].

Lemma 2.6. *Let $a = \frac{x+\sqrt{x^2-4}}{2}$, $b = \frac{x-\sqrt{x^2-4}}{2}$. Then*

$$\Phi(P_n) = \frac{1}{\sqrt{x^2-4}}(a^{n+1} - b^{n+1}).$$

Proof. We prove this result by induction on n. If $n = 1$ or $n = 2$, the result is obvious. Suppose that the result holds for $n \geqslant 2$. Since by definition we have $a^2 - xa + 1 = 0$ and $b^2 - xb + 1 = 0$. We have from Lemma 2.1 that

$$\begin{aligned}\Phi(P_{n+1}) &= x\Phi(P_n) - \Phi(P_{n-1}) \\ &= \frac{1}{\sqrt{x^2-4}}[x(a^{n+1} - b^{n+1}) - (a^n - b^n)] \\ &= \frac{1}{\sqrt{x^2-4}}[a^n(xa-1) - b^n(xb-1)] \\ &= \frac{1}{\sqrt{x^2-4}}(a^{n+2} - b^{n+2}).\end{aligned}$$

The result follows. □

Lemma 2.7 [10]. *Let $e = uv$ be an edge of a tree T each of the vertices u and v has degree at least two. Let T' be a tree obtained from T by deleting the edge e, identifying the vertices u and v (suppose that the new vertex is still denoted by u), and then attaching a new pendant edge uw at u. Then $\lambda_1(T') > \lambda_1(T)$.*

By applying Lemma 2.7 several times, we can further obtain the following.

Corollary 2.8. *Let u be a vertex of a tree T and $uv_1, uv_2, \ldots, uv_k$ are edges of T. Let T_i be the connected component (which is of course a subtree of T) of $T - uv_i$ containing the vertex v_i $(i = 1, \ldots, k)$. Write $|V(T_i)| = n_i$ and $n^* = \sum_{i=1}^{k} n_i$. Let $\widehat{T}$ be the tree obtained from the subgraph of T induced by the vertex subset $V(T) \setminus \bigcup_{i=1}^{k} V(T_i)$ by attaching n^* new pendant edges at the vertex u. Then we have $\lambda_1(T) \leqslant \lambda_1(\widehat{T})$, with equality if and only if $\widehat{T} = T$.*

An internal path of a graph G is a sequence of vertices $v_1, v_2, \ldots, v_k$ with $k \geqslant 2$ such that:

(1) the vertices in the sequence are distinct (except possibly $v_1 = v_k$);
(2) v_i is adjacent to v_{i+1} $(i = 1, 2, \ldots, k-1)$;
(3) the vertex degrees $d(v_i)$ satisfy $d(v_1) \geqslant 3, d(v_2) = \cdots = d(v_{k-1}) = 2$ (unless $k = 2$) and $d(v_k) \geqslant 3$.

Let W_n be the tree on n vertices obtained from a path P_{n-4} (of length $n - 5$) by attaching two new pendant edges to each end vertex of P_{n-4}, respectively. In [5], Hoffman and Smith obtained the following result:

Lemma 2.9. *Suppose that $G \neq W_n$ and uv is an edge on an internal path of G. Let G_{uv} be the graph obtained from G by the subdivision of the edge uv (i.e., by deleting the edge uv, adding a new vertex w and two new edges uw and wv). Then $\lambda_1(G_{uv}) < \lambda_1(G)$.*

Lemma 2.10 [4]. *Let w and v be two distinct vertices in a connected graph G and suppose that s new pendant edges $\{ww_1, ww_2, \ldots, ww_s\}$ are attached to G at w and t new pendant edges $\{vv_1, vv_2, \ldots, vv_t\}$ are attached to G at v to form a new graph $F_{s,t}$. Then either*

$$\lambda_1(F_{s+i,t-i}) > \lambda_1(F_{s,t}) \quad \text{for all } 1 \leqslant i \leqslant t,$$

or

$$\lambda_1(F_{s-i,t+i}) > \lambda_1(F_{s,t}) \quad \text{for all } 1 \leqslant i \leqslant s.$$

3. Main results

In this section, we will give the first $\left\lfloor \frac{d}{2} \right\rfloor + 1$ spectral radii of trees in the set $T_{(n,d)}$ $(3 \leqslant d \leqslant n-4)$. For this purpose, we first introduce several subsets ($T'_{(n,d)}$, $T^*_{(n,d)}$ and $\widetilde{T}_{(n,d)}$) of the set T of trees.

Let n, d, i be integers with $2 \leqslant i \leqslant d \leqslant n-2$. Let $T_{(n,d)}(i)$ be the tree on n vertices (with diameter d) obtained from a path $P_{d+1} : v_1 v_2 \cdots v_d v_{d+1}$ (of length d)

by attaching $n-d-1$ new pendant edges $v_iv_{d+2}, v_iv_{d+3}, \ldots, v_iv_n$ to the vertex v_i (see Fig. 1). It is easy to see that

$$T_{(n,d)}(i) = T_{(n,d)}(d+2-i) \quad (2 \leqslant i \leqslant d).$$

Let

$$T'_{(n,d)} = \{T_{(n,d)}(i) : \ i = 2, 3, \ldots, d\}$$

be a subset of $T_{(n,d)}$ and let

$$T' = T_{(n,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right).$$

Let $T^*_{(n,d)}(i)$ $(3 \leqslant i \leqslant d-1)$ be the tree on n vertices (with diameter d) obtained from a path $P_{d+1} : v_1v_2 \cdots v_dv_{d+1}$ (of length d) by attaching a new path P_3: $v_iv_{d+2}v_{d+3}$ (of length 2) and $n-d-3$ new pendant edges $v_iv_{d+4}, v_iv_{d+5}, \ldots, v_iv_n$ to the vertex v_i, respectively (see Fig. 1). Then it is easy to see that

$$T^*_{(n,d)}(i) = T^*_{(n,d)}(d+2-i) \quad (3 \leqslant i \leqslant d-1).$$

Let

$$T^*_{(n,d)} = \{T^*_{(n,d)}(i) : \ i = 3, 4, \ldots, d-1\}$$

be a subset of $T_{(n,d)}$ and let

$$T^* = T^*_{(n,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right).$$

Let $\widetilde{T}_{(n,d)}(i)$ be the tree on n vertices (with diameter d) obtained from a path $P_{d+1} : v_1v_2 \cdots v_i \cdots v_dv_{d+1}$ (of length d) and a star $K_{1,n-d-2}$ by addition of an edge joining the vertex v_i $(3 \leqslant i \leqslant d-1)$ of P_{d+1} and the center v_{d+2} of the star $K_{1,n-d-2}$ (see Fig. 2). Then it is easy to see that

$$\widetilde{T}_{(n,d)}(i) = \widetilde{T}_{(n,d)}(d-i+2) \quad \text{for } 3 \leqslant i \leqslant d-1.$$

Let

$$\widetilde{T}_{(n,d)} = \{\widetilde{T}_{(n,d)}(i) : i = 3, 4, \ldots, d-1\}$$

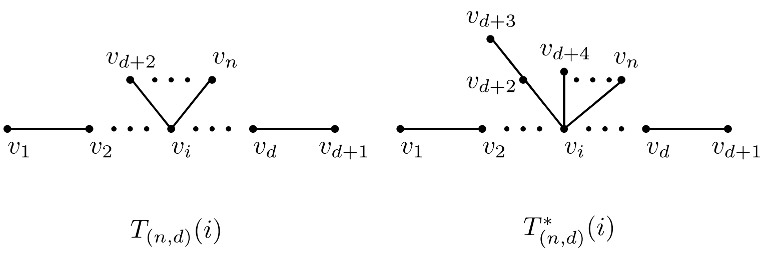

Fig. 1. $T_{(n,d)}(i)$ and $T^*_{(n,d)}(i)$.

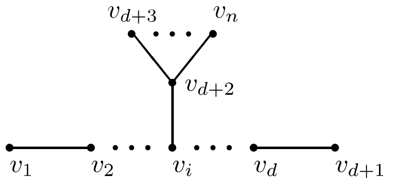

Fig. 2. $\widetilde{T}_{(n,d)}(i)$.

be a subset of $T_{(n,d)}$ and let

$$\widetilde{T} = \widetilde{T}_{(n,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right).$$

Lemma 3.1. *For any* $2 \leqslant j < i \leqslant \left\lfloor \frac{d}{2} \right\rfloor + 1$, *we have* $\lambda_1(T_{(n,d)}(i)) > \lambda_1(T_{(n,d)}(j))$. *Therefore, for any tree* $T \in T'_{(n,d)}$, $\lambda_1(T) \leqslant \lambda_1(T')$, *with equality up to isomorphism if and only if* $T \cong T'$.

Proof. In order to obtain the desired result, we only need to prove the case $j = i - 1$. Take $v = v_i$ and take the two paths $P = v_1 v_2 \cdots v_i$ and $Q = v_i v_{i+1} \cdots v_{d+1}$ in Lemma 2.4, we obtain the desired result. □

Corollary 3.2. *The first* $\left\lfloor \frac{d}{2} \right\rfloor$ *spectral radii of trees in the set* $T_{(n,d)}$ *with* $n = d + 2$ *are as follows*:

$$T' = T_{(n,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right), T_{(n,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor\right), \ldots, T_{(n,d)}(3), T_{(n,d)}(2).$$

By similar reasoning as above, we obtain the following two results:

Lemma 3.3. *For any* $3 \leqslant j < i \leqslant \left\lfloor \frac{d}{2} \right\rfloor + 1$, *we have* $\lambda_1(T^*_{(n,d)}(i)) > \lambda_1(T^*_{(n,d)}(j))$. *Therefore, for any tree* $T \in T^*_{(n,d)}$, $\lambda_1(T) \leqslant \lambda_1(T^*)$, *with equality up to isomorphism if and only if* $T \cong T^*$.

Lemma 3.4. *For any* $3 \leqslant j < i \leqslant \left\lfloor \frac{d}{2} \right\rfloor + 1$, *we have* $\lambda_1(\widetilde{T}_{(n,d)}(i)) > \lambda_1(\widetilde{T}_{(n,d)}(j))$. *Therefore, for any tree* $T \in \widetilde{T}_{(n,d)}$, $\lambda_1(T) \leqslant \lambda_1(\widetilde{T})$, *with equality up to isomorphism if and only if* $T \cong \widetilde{T}$.

Lemma 3.5. *For* $d \geqslant 4$, *we have*

$$\lambda_1(T^*) > \lambda_1(\widetilde{T}).$$

Proof. From Lemma 2.2, we have

$$\Phi(\widetilde{T}) = \Phi\left(\widetilde{T}_{(n,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right)\right)$$

$$
\begin{aligned}
&= x\Phi\left(\widetilde{T} - v_{\left\lfloor \frac{d}{2}\right\rfloor+1}\right) - \Phi\left(\widetilde{T} - v_{\left\lfloor \frac{d}{2}\right\rfloor+1} - v_{\left\lfloor \frac{d}{2}\right\rfloor}\right)\\
&\quad -\Phi\left(\widetilde{T} - v_{\left\lfloor \frac{d}{2}\right\rfloor+1} - v_{\left\lfloor \frac{d}{2}\right\rfloor+2}\right) - \Phi\left(\widetilde{T} - v_{\left\lfloor \frac{d}{2}\right\rfloor+1} - v_{d+2}\right)\\
&= x^{n-d-2}(x^2-(n-d-2))\Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor}\right)\\
&\quad -x^{n-d-3}(x^2-(n-d-2))\Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor-1}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor}\right)\\
&\quad -x^{n-d-3}(x^2-(n-d-2))\Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor-1}\right)\\
&\quad -x^{n-d-2}\Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor}\right),
\end{aligned}
$$

while,

$$
\begin{aligned}
\Phi(T^*) &= \Phi\left(T^*_{(n,d)}\left(\left\lfloor \frac{d}{2}\right\rfloor+1\right)\right)\\
&= x\Phi\left(T^* - v_{\left\lfloor \frac{d}{2}\right\rfloor+1}\right) - \Phi\left(T^* - v_{\left\lfloor \frac{d}{2}\right\rfloor+1} - v_{\left\lfloor \frac{d}{2}\right\rfloor}\right)\\
&\quad -\Phi\left(T^* - v_{\left\lfloor \frac{d}{2}\right\rfloor+1} - v_{\left\lfloor \frac{d}{2}\right\rfloor+2}\right) - \Phi\left(T^* - v_{\left\lfloor \frac{d}{2}\right\rfloor+1} - v_{d+2}\right)\\
&\quad -(n-d-3)\Phi\left(T^* - v_{\left\lfloor \frac{d}{2}\right\rfloor+1} - v_{d+4}\right)\\
&= x^{n-d-2}(x^2-1)\Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor}\right)\\
&\quad -x^{n-d-3}(x^2-1)\Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor-1}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor}\right)\\
&\quad -x^{n-d-3}(x^2-1)\Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor-1}\right)\\
&\quad -x^{n-d-2}\Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor}\right)\\
&\quad -(n-d-3)x^{n-d-4}(x^2-1)\Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor}\right).
\end{aligned}
$$

Then, for $x \geqslant 2$, we have

$$
\begin{aligned}
&\Phi(\widetilde{T}) - \Phi(T^*)\\
&= (d+3-n)x^{n-d-2}\Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+(n-d-3)x^{n-d-3}\Phi\left(P_{\lfloor\frac{d}{2}\rfloor-1}\right)\Phi\left(P_{d-\lfloor\frac{d}{2}\rfloor}\right)\\
&+(n-d-3)x^{n-d-3}\Phi\left(P_{\lfloor\frac{d}{2}\rfloor}\right)\Phi\left(P_{d-\lfloor\frac{d}{2}\rfloor-1}\right)\\
&+(n-d-3)x^{n-d-4}(x^2-1)\Phi\left(P_{\lfloor\frac{d}{2}\rfloor}\right)\Phi\left(P_{d-\lfloor\frac{d}{2}\rfloor}\right)\\
=\,&(n-d-3)x^{n-d-3}\Phi\left(P_{\lfloor\frac{d}{2}\rfloor-1}\right)\Phi\left(P_{d-\lfloor\frac{d}{2}\rfloor}\right)\\
&+(n-d-3)x^{n-d-3}\Phi\left(P_{\lfloor\frac{d}{2}\rfloor}\right)\Phi\left(P_{d-\lfloor\frac{d}{2}\rfloor-1}\right)\\
&-(n-d-3)x^{n-d-4}\Phi\left(P_{\lfloor\frac{d}{2}\rfloor}\right)\Phi\left(P_{d-\lfloor\frac{d}{2}\rfloor}\right)\\
=\,&(n-d-3)x^{n-d-4}\left[x\Phi\left(P_{\lfloor\frac{d}{2}\rfloor-1}\right)\Phi\left(P_{d-\lfloor\frac{d}{2}\rfloor}\right)\right.\\
&\left.+x\Phi\left(P_{\lfloor\frac{d}{2}\rfloor}\right)\Phi\left(P_{d-\lfloor\frac{d}{2}\rfloor-1}\right)-\Phi\left(P_{\lfloor\frac{d}{2}\rfloor}\right)\Phi\left(P_{d-\lfloor\frac{d}{2}\rfloor}\right)\right]\\
=\,&(n-d-3)x^{n-d-4}\left[x\Phi\left(P_{\lfloor\frac{d}{2}\rfloor}\right)\Phi\left(P_{d-\lfloor\frac{d}{2}\rfloor-1}\right)\right.\\
&\left.+\Phi\left(P_{\lfloor\frac{d}{2}\rfloor-2}\right)\Phi\left(P_{d-\lfloor\frac{d}{2}\rfloor}\right)\right]>0.
\end{aligned}
$$

We have $\lambda_1(T^*) > \lambda_1(\widetilde{T})$. The proof is complete. $\square$

Let n, d, i, j be integers with $i \neq j$ and $2 \leqslant i, j \leqslant d \leqslant n-3$. Next, we introduce another subset $T''_{(n,d)}$ of the set $T_{(n,d)}$ of trees. Let $T_{(n,d)}(i, j)$ be the tree on n vertices (with diameter d) obtained from a path $P_{d+1} : v_1v_2 \cdots v_dv_{d+1}$ (of length d) by attaching $n-d-2$ new pendant edges $v_iv_{d+2}, v_iv_{d+3}, \ldots, v_iv_{n-1}$ to v_i and a new pendant edge v_jv_n to v_j, respectively (see Fig. 3). It is obvious that

$$T_{(n,d)}(i, j) = T_{(n,d)}(d+2-i, d+2-j) \quad (2 \leqslant i, j \leqslant d).$$

So without loss of generality, we may always assume in the following that $i < j$ for $T_{(n,d)}(i, j)$. Let

$$T''_{(n,d)} = \{T_{(n,d)}(i, j) : 2 \leqslant i < j \leqslant d\}$$

be a subset of $T_{(n,d)}$ and let

$$T'' = T_{(n,d)}\left(\left\lfloor\frac{d}{2}\right\rfloor+1, \left\lfloor\frac{d}{2}\right\rfloor+2\right).$$

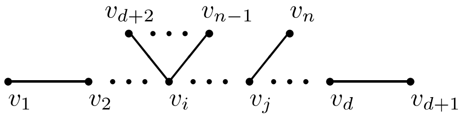

Fig. 3. $T_{(n,d)}(i, j)$

From the definition of $T_{(n,d)}(i, j)$, it is easy to see that $W_n = T_{(n,d)}(2, d)$ when $d = n - 3$ (where W_n is defined in Section 2).

Lemma 3.6. *For any* $2 \leqslant i < j \leqslant d$, *we have*

$$\lambda_1(T_{(n,d)}(i, j)) \leqslant \lambda_1(T'').$$

For $n \geqslant d + 4$, *equality holds if and only if* $i = \left\lfloor \frac{d}{2} \right\rfloor + 1$ *and* $j = \left\lfloor \frac{d}{2} \right\rfloor + 2$.

Proof. If $n = 6$, the result is obvious. Now, we assume that $n \geqslant 7$. If $T_{(n,d)}(i, j) = W_n$, since $\lambda_1(W_n) = 2$ for $n \geqslant 6$ (see, [1, p. 77]), we conclude from Lemma 2.3 that $\lambda_1(T'') > \lambda_1(T_{(6,3)}(2, 3)) = \lambda_1(W_n)$ for $n \geqslant 7$. So, we can further assume that $T_{(n,d)}(i, j) \neq W_n$. Firstly, we prove that

$$\lambda_1(T_{(n,d)}(i, j)) < \lambda_1(T_{(n,d)}(i, j - 1)) \quad (\text{for } j > i + 1).$$

Let $G = T_{(n-1,d-1)}(i, j - 1)$. Then G is a proper subgraph of $T_{(n,d)}(i, j - 1)$. Also, the graph $T_{(n,d)}(i, j)$ can be obtained from G by the subdivision of the edge $v_i v_{i+1}$ in an internal path $v_i v_{i+1} \cdots v_{j-1}$ of G. So by using Lemma 2.9 (since $T_{(n,d)}(i, j) \neq W_n$ implies $G \neq W_{n-1}$) and Lemma 2.3, we have

$$\lambda_1(T_{(n,d)}(i, j)) < \lambda_1(G) < \lambda_1(T_{(n,d)}(i, j - 1)) \quad (\text{for } j > i + 1).$$

Thus by induction on $j - i$, we finally have

$$\lambda_1(T_{(n,d)}(i, j)) < \lambda_1(T_{(n,d)}(i, i + 1)) \quad (\text{for } j > i + 1). \tag{3.1}$$

If d is odd, then by using Lemma 2.5 for two vertices v_i, v_{i+1} and the two paths $v_i v_{i-1} \cdots v_1$ and $v_{i+1} \cdots v_{d+1}$, we have

$$\lambda_1(T_{(n,d)}(i, i + 1)) < \lambda(T'') \quad \left(i \neq \left\lfloor \frac{d}{2} \right\rfloor + 1\right). \tag{3.2}$$

So the result follows from (3.1) and (3.2).

If d is even, then by using Lemma 2.5 for two vertices v_i, v_{i+1} and the two paths $v_i v_{i-1} \cdots v_1$ and $v_{i+1} \cdots v_{d+1}$, we can only have

$$\lambda_1(T_{(n,d)}(i, i + 1)) < \max\left\{\lambda_1(T''), \lambda_1\left(T_{(n,d)}\left(\frac{d}{2}, \frac{d}{2} + 1\right)\right)\right\}. \tag{3.3}$$

If $n = d + 3$, then we obviously have

$$T'' = T_{(n,d)}\left(\frac{d}{2}, \frac{d}{2} + 1\right).$$

So the result follows from (3.1) and (3.3). So we can assume $n \geqslant d+4$ and we need only to prove that

$$\lambda_1(T'') > \lambda_1\left(T_{(n,d)}\left(\frac{d}{2}, \frac{d}{2}+1\right)\right) \quad (d \text{ is even and } n \geqslant d+4).$$

From Lemma 2.2, we have

$$\begin{aligned}
\Phi(T'') &= x\Phi\left(T'' - v_{\frac{d}{2}+1}\right) - \Phi\left(T'' - v_{\frac{d}{2}+1} - v_{\frac{d}{2}}\right) \\
&\quad -\Phi\left(T'' - v_{\frac{d}{2}+1} - v_{\frac{d}{2}+2}\right) - \sum_{i=d+2}^{n-1} \Phi\left(T'' - v_{\frac{d}{2}+1} - v_i\right) \\
&= x^{n-d-1}\Phi\left(P_{\frac{d}{2}}\right)\Phi\left(P_{\frac{d}{2}+1}\right) - x^{n-d-2}\Phi\left(P_{\frac{d}{2}-1}\right)\Phi\left(P_{\frac{d}{2}+1}\right) \\
&\quad -x^{n-d-1}\Phi\left(P_{\frac{d}{2}}\right)\Phi\left(P_{\frac{d}{2}-1}\right) - (n-d-2)x^{n-d-3}\Phi\left(P_{\frac{d}{2}}\right)\Phi\left(P_{\frac{d}{2}+1}\right).
\end{aligned}$$

We also have

$$\begin{aligned}
&\Phi\left(T_{(n,d)}\left(\frac{d}{2}, \frac{d}{2}+1\right)\right) \\
&\quad = x\Phi\left(T_{(n,d)}\left(\frac{d}{2}, \frac{d}{2}+1\right) - v_{\frac{d}{2}}\right) \\
&\qquad -\Phi\left(T_{(n,d)}\left(\frac{d}{2}, \frac{d}{2}+1\right) - v_{\frac{d}{2}} - v_{\frac{d}{2}+1}\right) \\
&\qquad -\Phi\left(T_{(n,d)}\left(\frac{d}{2}, \frac{d}{2}+1\right) - v_{\frac{d}{2}} - v_{\frac{d}{2}-1}\right) \\
&\qquad - \sum_{i=d+2}^{n-1} \Phi\left(T_{(n,d)}\left(\frac{d}{2}, \frac{d}{2}+1\right) - v_{\frac{d}{2}} - v_i\right) \\
&\quad = x^{n-d-1}\Phi\left(P_{\frac{d}{2}-1}\right)\Phi\left(P_{\frac{d}{2}+2}\right) - x^{n-d-1}\Phi\left(P_{\frac{d}{2}-1}\right)\Phi\left(P_{\frac{d}{2}}\right) \\
&\qquad -x^{n-d-2}\Phi\left(P_{\frac{d}{2}+2}\right)\Phi\left(P_{\frac{d}{2}-2}\right) \\
&\qquad -(n-d-2)x^{n-d-3}\Phi\left(P_{\frac{d}{2}-1}\right)\Phi\left(P_{\frac{d}{2}+2}\right).
\end{aligned}$$

Thus, by using the formula for $\Phi(P_n)$ given in Lemma 2.6, we have

$$\begin{aligned}
&\Phi(T'') - \Phi\left(T_{(n,d)}\left(\frac{d}{2}, \frac{d}{2}+1\right)\right) \\
&\quad = x^{n-d-1}\left(\Phi\left(P_{\frac{d}{2}}\right)\Phi\left(P_{\frac{d}{2}+1}\right) - \Phi\left(P_{\frac{d}{2}-1}\right)\Phi\left(P_{\frac{d}{2}+2}\right)\right) \\
&\qquad +x^{n-d-2}\left(\Phi\left(P_{\frac{d}{2}+2}\right)\Phi\left(P_{\frac{d}{2}-2}\right) - \Phi\left(P_{\frac{d}{2}-1}\right)\Phi\left(P_{\frac{d}{2}+1}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+(n-d-2)x^{n-d-3}\left(\Phi\left(P_{\frac{d}{2}-1}\right)\Phi\left(P_{\frac{d}{2}+2}\right)-\Phi\left(P_{\frac{d}{2}}\right)\Phi\left(P_{\frac{d}{2}+1}\right)\right)\\
&=\frac{x^{n-d-1}}{x^2-4}(a^3+b^3-a-b)+\frac{x^{n-d-2}}{x^2-4}(a^2+b^2-a^4-b^4)\\
&\quad+\frac{(n-d-2)x^{n-d-3}}{x^2-4}(a+b-a^3-b^3)\\
&=\frac{x^{n-d-3}}{x^2-4}[x^2(a^3+b^3-a-b)+x(a^2+b^2-a^4-b^4)\\
&\qquad+(n-d-2)(a+b-a^3-b^3)]\\
&=\frac{x^{n-d-3}}{x^2-4}[(a-b)^2(a+b)(x^2-(n-d-2))\\
&\qquad-(a-b)^2(b^2+a^2+1)x]\\
&=x^{n-d-2}(d+3-n),
\end{aligned}
$$

which is negative for $x > 0$ and $n \geqslant d+4$.

So we have

$$\lambda_1(T'') > \lambda_1\left(T_{(n,d)}\left(\frac{d}{2}, \frac{d}{2}+1\right)\right)$$

for d is even and $n \geqslant d+4$, and the result follows from (3.1) and (3.3). $\square$

Lemma 3.7. *For $n \geqslant d+3 \geqslant 7$, we have*

$$\lambda_1(T'') > \lambda_1(T^*).$$

Proof. We have from Lemma 2.1 that

$$\Phi(T'') = \Phi\left(T'' - v_{\lfloor \frac{d}{2} \rfloor+2}v_n\right) - \Phi\left(T'' - v_{\lfloor \frac{d}{2} \rfloor+2} - v_n\right),$$

while

$$\Phi(T^*) = \Phi(T^* - v_{d+2}v_{d+3}) - \Phi(T^* - v_{d+2} - v_{d+3}).$$

Since

$$T'' - v_{\lfloor \frac{d}{2} \rfloor+2}v_n = T_{(n-1,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right) = T^* - v_{d+2}v_{d+3}$$

and $T'' - v_{\lfloor \frac{d}{2} \rfloor+2} - v_n$ is a proper spanning subgraph of $T^* - v_{d+2} - v_{d+3}$.

We have from Lemma 2.3 that

$$\Phi(T'') < \Phi(T^*)$$

for $x \geqslant \lambda_1(T^* - v_{d+2} - v_{d+3})$, thus for $x \geqslant \lambda_1(T^*)$.

So, we have

$$\lambda_1(T'') > \lambda_1(T^*). \qquad \square$$

Lemma 3.8. *For any tree $T \in T_{(n,d)} \backslash \{T'_{(n,d)} \cup T''\}$ with $n \geqslant d+4$ and $d \geqslant 3$, we have $\lambda_1(T) < \lambda_1(T'')$.*

Proof. Since $T \in T_{(n,d)}$, without loss of generality, we may assume that there exists a path, say $P_{d+1} : v_1 v_2 \cdots v_d v_{d+1}$, of length d of T and $d(v_1) = d(v_{d+1}) = 1$. Since $n \geqslant d+4$, there exists at least one vertex, say v_i $(2 \leqslant i \leqslant d)$, such that $d(v_i) \geqslant 3$. Now, we distinguish the following two cases:

Case 1. There exist at least two distinct vertices of $v_2, v_3, \ldots, v_d$ (say, $v_{i_1}, \ldots, v_{i_k}$ with $k \geqslant 2$) having degree at least 3. By applying Corollary 2.8 to each vertex v_{i_j} of T, $j = 1, \ldots, k$, we can obtain a tree T_1 which is obtained from P_{d+1} by attaching suitable stars at the vertices $v_{i_1}, \ldots, v_{i_k}$ such that

$$\lambda_1(T) \leqslant \lambda_1(T_1) \tag{3.4}$$

with equality if and only if $T_1 \cong T$.

By applying Lemma 2.10 several times to T_1, we can further obtain a tree T_2 which is of the similar type as T_1 in the special case $k = 2$ such that

$$\lambda_1(T_1) \leqslant \lambda_1(T_2) \tag{3.5}$$

with equality if and only if $T_2 \cong T_1$.

Finally, applying Lemma 2.10 once more to T_2, we can obtain a tree T_3 in $T''_{(n,d)}$ such that

$$\lambda_1(T_2) \leqslant \lambda_1(T_3) \tag{3.6}$$

with equality if and only if $T_3 \cong T_2$.

Now from Lemma 3.6 and the hypothesis $n \geqslant d+4$, we have

$$\lambda_1(T_3) \leqslant \lambda_1(T'') \tag{3.7}$$

with equality if and only if $T_3 \cong T''$.

Combining (3.4)–(3.7), we have $\lambda_1(T) \leqslant \lambda_1(T'')$, with equality if and only if $T \cong T''$. But $T \ncong T''$ by hypothesis, so we have $\lambda_1(T) < \lambda_1(T'')$ as desired.

Case 2. There exists exactly one vertex in $\{v_2, \ldots, v_d\}$, say v_i $(3 \leqslant i \leqslant d-1)$, having degree $d(v_i) \geqslant 3$. Let $w_1, \ldots, w_s$ be all the vertices outside the path P_{d+1} which are adjacent to v_i, among which $w_1, \ldots, w_r$ are of degree at least two and $w_{r+1}, \ldots, w_s$ are pendant vertices. Then $r \geqslant 1$ since T is not in $T'_{(n,d)}$ by hypothesis. Further, we have $d \geqslant 4$.

By using Corollary 2.8 to each vertex w_j $(j = 1, \ldots, r)$ of T, we can obtain a tree T_1 obtained from $P_{d+1} + v_i w_1 + \cdots + v_i w_s$ by attaching suitable stars at the vertices $w_1, \ldots, w_r$ such that

$$\lambda_1(T) \leqslant \lambda_1(T_1). \tag{3.8}$$

Now by applying Lemma 2.10 several times to T_1, we can obtain a tree T_2 either in $T^*_{(n,d)}$ or in $\widetilde{T}_{(n,d)}$ such that

$$\lambda_1(T_1) \leqslant \lambda_1(T_2). \tag{3.9}$$

If $T_2 \in T^*_{(n,d)}$, then from Lemmas 3.3, 3.7 and (3.8), (3.9) we have

$$\lambda_1(T) \leqslant \lambda_1(T_1) \leqslant \lambda_1(T_2) \leqslant \lambda_1(T^*) < \lambda_1(T'')$$

as desired.

If $T_2 \in \widetilde{T}_{(n,d)}$, then from Lemmas 3.4, 3.5, 3.7 and (3.8), (3.9) we have

$$\lambda_1(T) \leqslant \lambda_1(T_1) \leqslant \lambda_1(T_2) \leqslant \lambda_1(\widetilde{T}) < \lambda_1(T^*) < \lambda_1(T'').$$

This completes the proof. □

Now, we can give the main result of this paper.

Theorem 3.9. *The first* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *spectral radii of trees in the set* $T_{(n,d)}$ *with* $n \geqslant d+4$ *and* $d \geqslant 3$ *are as follows*:

$$T' = T_{(n,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right), T_{(n,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor\right), \ldots, T_{(n,d)}(3), T_{(n,d)}(2), T''.$$

Proof. From Lemma 3.1, we have

$$\lambda_1(T') > \lambda_1\left(T_{(n,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor\right)\right) > \cdots > \lambda_1(T_{(n,d)}(3)) > \lambda_1(T_{(n,d)}(2)).$$

So from Lemma 3.8, we only need to prove that

$$\lambda_1(T_{(n,d)}(2)) > \lambda_1(T'')$$

for $n \geqslant d+4 \geqslant 7$.

Since $n \geqslant d+4$ and $\lambda_1(T_{(7,3)}(2)) \approx 2.288 > \sqrt{5} \approx 2.236$, we have from Lemma 2.3 that $\lambda_1(T_{(n,d)}(2)) > \sqrt{5}$, for $n \geqslant 7$.

From Lemma 2.1, we have

$$\begin{aligned}\Phi(T'') = {} & x^{n-d-1}\Phi(P_{d+1}) - (n-d-2)x^{n-d-3}\Phi\left(P_{\left\lfloor \frac{d}{2} \right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2} \right\rfloor+1}\right) \\ & -x^{n-d-2}\Phi\left(P_{\left\lfloor \frac{d}{2} \right\rfloor+1}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2} \right\rfloor-1}\right),\end{aligned} \tag{3.10}$$

while

$$\Phi(T_{(n,d)}(2)) = x^{n-d-1}\Phi(P_{d+1}) - (n-d-1)x^{n-d-1}\Phi(P_{d-1}).$$

Then, we have

$$\begin{aligned}&\Phi(T'') - \Phi(T_{(n,d)}(2)) \\ &\quad = (n-d-1)x^{n-d-1}\Phi(P_{d-1}) - x^{n-d-2}\Phi\left(P_{\left\lfloor \frac{d}{2} \right\rfloor+1}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2} \right\rfloor-1}\right) \\ &\qquad -(n-d-2)x^{n-d-3}\Phi\left(P_{\left\lfloor \frac{d}{2} \right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2} \right\rfloor+1}\right)\end{aligned}$$

$$\begin{aligned}&= (n-d-2)x^{n-d-3}\left[x^2\Phi(P_{d-1}) - \Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor+1}\right)\right]\\ &\quad + x^{n-d-2}\left[x\Phi(P_{d-1}) - \Phi\left(P_{\left\lfloor \frac{d}{2}\right\rfloor+1}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2}\right\rfloor-1}\right)\right].\end{aligned} \tag{3.11}$$

In the following, we prove that for $x > \sqrt{5}$, the right side of (3.11) is positive. Note that for $x > \frac{3\sqrt{2}}{2} \approx 2.121$, and $a = \frac{x+\sqrt{x^2-4}}{2}$, we have

$$\begin{aligned}x^2\sqrt{x^2-4} - a^3 &= \frac{x^2\sqrt{x^2-4} - x^3 + 3x + \sqrt{x^2-4}}{2}\\ &> \frac{x(x\sqrt{x^2-4} - x^2 + 3)}{2} > 0.\end{aligned}$$

So the right side of (3.11)

$$\begin{aligned}&= \frac{(n-d-2)x^{n-d-3}}{x^2-4}\left[x^2\sqrt{x^2-4}(a^d - b^d) - \left(a^{\left\lfloor \frac{d}{2}\right\rfloor+1} - b^{\left\lfloor \frac{d}{2}\right\rfloor+1}\right)\right.\\ &\qquad\left.\times\left(a^{d-\left\lfloor \frac{d}{2}\right\rfloor+2} - b^{d-\left\lfloor \frac{d}{2}\right\rfloor+2}\right)\right]\\ &\quad + \frac{x^{n-d-2}}{x^2-4}\left[x\sqrt{x^2-4}(a^d - b^d) - \left(a^{\left\lfloor \frac{d}{2}\right\rfloor+2} - b^{\left\lfloor \frac{d}{2}\right\rfloor+2}\right)\right.\\ &\qquad\left.\times\left(a^{d-\left\lfloor \frac{d}{2}\right\rfloor} - b^{d-\left\lfloor \frac{d}{2}\right\rfloor}\right)\right]\\ &= \frac{(n-d-2)x^{n-d-3}}{x^2-4}\left[x^2\sqrt{x^2-4}(a^d - b^d) - a^{d+3} - b^{d+3}\right.\\ &\qquad\left. + a^{d-2\left\lfloor \frac{d}{2}\right\rfloor+1} + b^{d-2\left\lfloor \frac{d}{2}\right\rfloor+1}\right]\\ &\quad + \frac{x^{n-d-2}}{x^2-4}\left[x\sqrt{x^2-4}(a^d - b^d) - a^{d+2} - b^{d+2}\right.\\ &\qquad\left. + a^{2\left\lfloor \frac{d}{2}\right\rfloor-d+2} + b^{2\left\lfloor \frac{d}{2}\right\rfloor-d+2}\right]\\ &> \frac{(n-d-2)x^{n-d-3}}{x^2-4}\left[a^d\left(x^2\sqrt{x^2-4} - a^3\right)\right.\\ &\qquad\left. - x^2\sqrt{x^2-4}b^d + a^{d-2\left\lfloor \frac{d}{2}\right\rfloor+1}\right]\\ &\quad + \frac{x^{n-d-2}}{x^2-4}\left[a^d\left(x\sqrt{x^2-4} - a^2\right) - x\sqrt{x^2-4}b^d + a^{2\left\lfloor \frac{d}{2}\right\rfloor-d+2}\right]\\ &> \frac{(n-d-2)x^{n-d-3}}{x^2-4}(a^d - b^d)\left(x^2\sqrt{x^2-4} - a^3\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{x^{n-d-2}}{x^2-4}(a^d-b^d)\left(x\sqrt{x^2-4}-a^2\right)\\
&=\frac{x^{n-d-3}(a^d-b^d)}{x^2-4}\left[(n-d-2)\left(x^2\sqrt{x^2-4}-a^3\right)+x^2\sqrt{x^2-4}-xa^2\right]\\
&>\frac{x^{n-d-3}(a^d-b^d)}{x^2-4}\left(\frac{2\left(x^2\sqrt{x^2-4}-x^3+3x+\sqrt{x^2-4}\right)}{2}\right.\\
&\qquad\left.+\frac{x^2\sqrt{x^2-4}-x^3+2x}{2}\right)\\
&=\frac{x^{n-d-3}(a^d-b^d)}{x^2-4}\cdot\frac{3x^2\sqrt{x^2-4}-3x^3+8x+2\sqrt{x^2-4}}{2}>0
\end{aligned}
$$

for $x > \sqrt{5}$.

Thus, we have for $x \geqslant \Phi(T_{(n,d)}(2)) > \sqrt{5}$,

$$\Phi(T'') > \Phi(T_{(n,d)}(2)).$$

So,

$$\lambda_1(T_{(n,d)}(2)) > \lambda_1(T'').$$

The proof is complete. $\square$

Finally, we settle the case $n = d + 3$.

Theorem 3.10. *The first $\left\lfloor \frac{d}{2} \right\rfloor - 1$ spectral radii of trees in the set $T_{(n,d)}$ with $n = d + 3$ and $d \geqslant 4$ are as follows*:

$$T' = T_{(n,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right), T_{(n,d)}\left(\left\lfloor \frac{d}{2} \right\rfloor\right), \ldots, T_{(n,d)}(3).$$

Proof. From Corollary 3.2 and Lemmas 3.6 and 3.7, we only need to prove that

$$\lambda_1(T_{(n,d)}(3)) > \lambda_1(T'')$$

for $n = d + 3$ and $d \geqslant 4$.

If $7 \leqslant n \leqslant 9$, it is easy to see that the result holds from the tables of the graph spectra in [1]. We now assume that $n \geqslant 10$. For $n = d + 3$, we have from Lemma 2.1 that

$$\Phi(T_{(n,d)}(3)) = x^2\Phi(P_{d+1}) - 2x\Phi(P_2)\Phi(P_{d-2}).$$

Thus, combining this with (3.10), we have

$$
\begin{aligned}
&\Phi(T_{(n,d)}(3)) - \Phi(T'')\\
&\quad= \Phi\left(P_{\left\lfloor \frac{d}{2} \right\rfloor}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2} \right\rfloor+1}\right) + x\Phi\left(P_{\left\lfloor \frac{d}{2} \right\rfloor+1}\right)\Phi\left(P_{d-\left\lfloor \frac{d}{2} \right\rfloor-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-2x\Phi(P_2)\Phi(P_{d-2})\\
&=\frac{1}{x^2-4}\Bigg[\left(a^{\lfloor\frac{d}{2}\rfloor+1}-b^{\lfloor\frac{d}{2}\rfloor+1}\right)\left(a^{d-\lfloor\frac{d}{2}\rfloor+2}-b^{d-\lfloor\frac{d}{2}\rfloor+2}\right)\\
&\qquad+x\left(a^{\lfloor\frac{d}{2}\rfloor+2}-b^{\lfloor\frac{d}{2}\rfloor+2}\right)\left(a^{d-\lfloor\frac{d}{2}\rfloor}-b^{d-\lfloor\frac{d}{2}\rfloor}\right)\\
&\qquad-2x(a^3-b^3)(a^{d-1}-b^{d-1})\Bigg]\\
&=\frac{1}{x^2-4}\Bigg[\left(a^{d+3}-b^{d+3}-a^{\lfloor\frac{d}{2}\rfloor+1}b^{d-\lfloor\frac{d}{2}\rfloor+2}-a^{d-\lfloor\frac{d}{2}\rfloor+2}b^{\lfloor\frac{d}{2}\rfloor+1}\right)\\
&\qquad+x\left(a^{d+2}+b^{d+2}-a^{\lfloor\frac{d}{2}\rfloor+2}b^{d-\lfloor\frac{d}{2}\rfloor}-a^{d-\lfloor\frac{d}{2}\rfloor}b^{\lfloor\frac{d}{2}\rfloor+2}\right)\\
&\qquad-2x(a^{d+2}+b^{d+2}-a^3b^{d-1}-a^{d-1}b^3)\Bigg]\\
&=\frac{1}{x^2-4}\Bigg[a^{d+2}(a-x)+b^{d+2}(b-x)-b^{d-2\lfloor\frac{d}{2}\rfloor+1}-a^{d-2\lfloor\frac{d}{2}\rfloor+1}\\
&\qquad-xa^{d-2\lfloor\frac{d}{2}\rfloor-2}-xb^{d-2\lfloor\frac{d}{2}\rfloor-2}+2xa^{d-4}+2xb^{d-4}\Bigg]\\
&=\frac{1}{x^2-4}\Bigg[-a^{d+1}-b^{d+1}-b^{d-2\lfloor\frac{d}{2}\rfloor+1}-a^{d-2\lfloor\frac{d}{2}\rfloor+1}-xa^{d-2\lfloor\frac{d}{2}\rfloor-2}\\
&\qquad-xb^{d-2\lfloor\frac{d}{2}\rfloor-2}+2xa^{d-4}+2xb^{d-4}\Bigg]\\
&<\frac{1}{x^2-4}(-a^{d+1}+2xa^{d-4})\\
&=\frac{-a^{d-4}}{2(x^2-4)}\left(x^5+x^4\sqrt{x^2-4}-5x^3+x-3x^2\sqrt{x^2-4}+\sqrt{x^2-4}\right)\\
&<\frac{-a^{d-4}}{2(x^2-4)}\left(x^3+x^2\sqrt{x^2-4}-5x-3\sqrt{x^2-4}\right)<0
\end{aligned}
$$

for $x \geqslant \lambda_1(T'') \geqslant \lambda_1(T_{(10,7)}(4,5)) \approx 2.127 > \frac{3\sqrt{2}}{2} \approx 2.121$.

Thus, we have

$$\lambda_1(T_{(n,d)}(3)) > \lambda_1(T'')$$

for $n = d + 3 \geqslant 7$. The proof is complete. □

Finally, we would like to point out that the following well-known result: "Among all the trees with n vertices, the star $K_{1,n-1}$ has the largest spectral radius." is now an easy consequence of our results. Indeed, if we simply denote the tree $T_{(n,d)}(\lfloor \frac{d}{2} \rfloor + 1)$ by $G(n, d)$, then by using Lemma 2.4 it is easy to show that

$$\lambda_1(G(n, d)) \leqslant \lambda_1(G(n, d - 1)) \quad (3 \leqslant d \leqslant n - 1).$$

So $G(n, 2) = K_{1,n-1}$ has the largest spectral radius.

References

[1] D.M. Cvetkovic, M. Doob, H. Sachs, Spectra of Graphs—Theory and Application. Deutscher Verlag der Wissenschaften—Academic Press, Berlin–New York, 1980; second ed. 1982; third ed., Johann Ambrosius Barth Verlag, Heidelberg–Leipzig, 1995.
[2] D.M. Cvetković, Some possible directions in further investigations of graph spectra, Algebraic Methods in Graph Theory, vol. 1, North-Holland, Amsterdam, 1981, pp. 47–67.
[3] A. Chang, Q. Huang, Ordering trees by their largest eigenvalues, Linear Algebra Appl. 370 (2003) 174–184.
[4] J.M. Guo, S.W. Tan, On the spectral radius of trees, Linear Algebra Appl. 329 (2001) 1–8.
[5] A.J. Hoffman, J.H. Smith, On the spectral radii of topologically equivalent graphs, in: M. Fiedler (Ed.), Recent Advances in Graph Theory, Academic Praha, 1975, pp. 273–281.
[6] A.J. Hoffman, On limit points on spectral radii of non-negative symmetric integral matrices, in: Y. Alavi et al., (Ed.), Lecture Notes Math., vol. 303, Springer-Verlag, Berlin–Heidelberg–New York, 1972, pp. 165–172.
[7] M. Hofmeister, On the two largest eigenvalues of trees, Linear Algebra Appl. 360 (1997) 43–59.
[8] Q. Li, K.Q. Feng, On the largest eigenvalue of graphs, Acta Math. Appl. Sinica 2 (1979) 167–175 (in Chinese).
[9] D. Stevanović, Bounding the largest eigenvalue of trees in terms of the largest vertex degree, Linear Algebra Appl. 360 (2003) 35–42.
[10] G.H. Xu, On the spectral radius of trees with perfect matchings, in: Combinatorics and Graph Theory, World Scientific, Singapore, 1997.